On supercyclicity of operators from a supercyclic semigroup

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Abstract

We show that for every supercyclic strongly continuous operator semigroup $\{T_t\}_{t\geqslant 0}$ acting on a complex $\mathcal{F}$-space, every T_t with $t>0$ is supercyclic. Moreover, the set of supercyclic vectors of each T_t with $t>0$ is exactly the set of supercyclic vectors of the entire semigroup.

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1 Introduction

Unless stated otherwise, all vector spaces in this article are over the field $\mathbb{K}$, being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers and all topological spaces *are assumed to be Hausdorff*. As usual, $\mathbb{Z}_+$ is the set of non-negative integers, $\mathbb{N}$ is the set of positive integers and $\mathbb{R}_+$ is the set of non-negative real numbers. The symbol $L(X)$ stands for the space of continuous linear operators on a topological vector space X, while X' is the space of continuous linear functionals on X. As usual, for $T\in L(X)$, the dual operator $T':X'\to X'$ is defined by the formula $T'f(x)=f(Tx)$ for $x\in X$ and $f\in X'$. Recall that an *affine* map T on a vector space X is a map of the shape $Tx=u+Sx$, where u is a fixed vector in X and $S:X\to X$ is linear. Clearly, T is continuous if and only if S is continuous. The symbol $A(X)$ stands for the space of continuous affine maps on a topological vector space X. An $\mathcal{F}$-space is a complete metrizable topological vector space. Recall that a family $\mathcal{F}=\{T_a\}_{a\in A}$ of continuous maps from a topological space X to a topological space Y is called *universal* if there is $x\in X$ for which $\{T_ax:a\in A\}$ is dense in Y and such an x is called a *universal element* for $\mathcal{F}$. We use the symbol $\mathcal{U}(\mathcal{F})$ for the set of universal elements for $\mathcal{F}$. If X is a topological space and $T:X\to X$ is a continuous map, then we say that $x\in X$ is *universal* for T if x is universal for the family $\{T^n:n\in\mathbb{Z}_+\}$. We denote the set of universal elements for T by $\mathcal{U}(T)$. A family $\mathcal{F}=\{T_t\}_{t\in\mathbb{R}_+}$ of continuous maps from a topological space X to itself is called a *semigroup* if $T_0=I$ and $T_{t+s}=T_tT_s$ for every $t,s\in\mathbb{R}_+$. We say that a semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ is *strongly continuous* if $t\mapsto T_tx$ is continuous as a map from $\mathbb{R}_+$ to X for every $x\in X$ and we say that $\{T_t\}_{t\in\mathbb{R}_+}$ is *jointly continuous* if $(t,x)\mapsto T_tx$ is continuous as a map from $\mathbb{R}_+\times X$ to X. If X is a topological vector space, we call a semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ a *linear semigroup* if $T_t\in L(X)$ for every $t\in\mathbb{R}_+$ and $\{T_t\}_{t\in\mathbb{R}_+}$ is called an *affine semigroup* if $T_t\in A(X)$ for every $t\in\mathbb{R}_+$. Recall that $T\in L(X)$ is called *hypercyclic* if $\mathcal{U}(T)\neq\varnothing$ and elements of $\mathcal{U}(T)$ are called *hypercyclic vectors*. A universal linear semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ is called *hypercyclic* and its universal elements are called *hypercyclic vectors* for $\{T_t\}_{t\in\mathbb{R}_+}$. If $T\in L(X)$, then universal elements of the family $\{zT^nx:z\in\mathbb{K},\ n\in\mathbb{Z}_+\}$ are called *supercyclic vectors* for T and T is called *supercyclic* if it has a supercyclic vector. Similarly, if $\{T_t\}_{t\in\mathbb{R}_+}$ is a linear semigroup, then a universal element of the family $\{zT_t:z\in\mathbb{K},\ t\in\mathbb{R}_+\}$ is called a *supercyclic vector* for $\{T_t\}_{t\in\mathbb{R}_+}$ and the semigroup is called *supercyclic* if it has a supercyclic vector.

Hypercyclicity and supercyclicity have been intensely studied during the last few decades, see [1] and references therein. Our concern is the relation between the supercyclicity of a linear semigroup and supercyclicity of the individual members of the semigroup. The hypercyclicity version of the question was treated by Conejero, Müller and Peris [3], who proved that for every

strongly continuous hypercyclic linear semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ on an $\mathcal{F}$-space, each T_s with $s>0$ is hypercyclic and $\mathcal{U}(T_s)=\mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$. Virtually the same proof works in the following much more general setting [1, Chapter 3].

Theorem A. *Let $\{T_t\}_{t\in\mathbb{R}_+}$ be a hypercyclic jointly continuous linear semigroup on any topological vector space X. Then each T_s with $s>0$ is hypercyclic and $\mathcal{U}(T_s)=\mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$.*

The stronger condition of joint continuity coincides with the strong continuity in the case when X is an $\mathcal{F}$-space due to a straightforward application of the Banach–Steinhaus theorem. The essential part of the proofs in [3, 1] does not really need linearity. It is based on a homotopy-type argument and goes through without any changes (under certain assumptions) for semigroups of non-linear maps. Recall that a topological space X is called *connected* if it has no subsets different from $\varnothing$ and X, which are closed and open and it is called *simply connected* if for any continuous map $f:\mathbb{T}\to X$, there is a continuous map $F:\mathbb{T}\times[0,1]\to X$ and $x_0\in X$ such that $F(z,0)=f(z)$ and $F(z,1)=x_0$ for any $z\in\mathbb{T}$. Next, X is called *locally path connected at $x\in X$* if for any neighborhood U of x, there is a neighborhood V of x such that for any $y\in V$, there is a continuous map $f:[0,1]\to X$ satisfying $f(0)=x$, $f(1)=y$ and $f([0,1])\subseteq U$. A space X is called *locally path connected* if it is locally path connected at every point. Just listing the conditions needed for the proof in [3, 1] to run smoothly, we get the following result.

Proposition 1.1. *Let X be a topological space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a jointly continuous semigroup on X such that*

1. *$\{T_tu: t\in[0,c]\}$ is nowhere dense in X for every $c>0$ and $u\in X$;*
2. *for every $c>0$ and $x\in\mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$, there is $Y_{c,x}\subseteq X$ such that $Y_{c,x}$ is connected, locally path connected, simply connected and $\{T_tx: t\in[0,c]\}\subseteq Y_{c,x}\subseteq\mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$.*

Then $\mathcal{U}(T_s)=\mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$ for every $s>0$.

The natural question whether the supercyclicity version of Theorem A holds was touched by Bernal-González and Grosse-Erdmann in [2]. They have produced the following example.

Example B. *Let X be a Banach space over $\mathbb{R}$, $\{T_t\}_{t\in\mathbb{R}_+}$ be a hypercyclic linear semigroup on X and $A_t\in L(\mathbb{R}^2)$ for $t\in\mathbb{R}_+$ be the linear operator with the matrix $A=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$. Then $\{A_t\oplus T_t\}_{t\in\mathbb{R}_+}$ is a supercyclic linear semigroup on $\mathbb{R}^2\times X$, while $A_t\oplus T_t$ is non-supercyclic whenever $\frac{t}{\pi}$ is rational.*

Example B shows that the natural supercyclicity version of Theorem A fails in the case $\mathbb{K}=\mathbb{R}$. In the complex case, the following partial result was obtained by Bayart and Matheron [1, p. 73].

Proposition C. *Let X be a complex topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a supercyclic jointly continuous linear semigroup on X such that $T_t-\lambda I$ has dense range for every $t>0$ and every $\lambda\in\mathbb{C}$. Then each T_t with $t>0$ is supercyclic. Moreover, the set of supercyclic vectors for T_t does not depend on the choice of $t>0$ and coincides with the set of supercyclic vectors of the entire semigroup.*

The argument in [1] is another adaptation of the proof in [3], however one can obtain the same result directly by considering the induced action on subsets of the projective space and applying Proposition 1.1. We will show that in the case $\mathbb{K}=\mathbb{C}$, the supercyclicity version of Theorem A holds without any additional assumptions.

Theorem 1.2. *Let X be a complex topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a supercyclic jointly continuous linear semigroup on X. Then each T_s with $s>0$ is supercyclic and the set of supercyclic vectors of T_s coincides with the set of supercyclic vectors of $\{T_t\}_{t\in\mathbb{R}_+}$.*

It turns out that any supercyclic jointly continuous linear semigroup on a complex topological vector X either satisfies conditions of Proposition C or has a closed invariant hyperplane Y. In the latter case the issue reduces to the following generalization of Theorem A to affine semigroups.

Theorem 1.3. *Let X be a topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a universal jointly continuous affine semigroup on X. Then each T_s with $s>0$ is universal and $\mathcal{U}(T_s)=\mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$.*

2 A dichotomy for supercyclic linear semigroups

An analogue of the following result for individual supercyclic operators is well-known [1].

Proposition 2.1. *Let X be a complex topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a supercyclic strongly continuous linear semigroup on X. Then either $(T_t-\lambda I)(X)$ is dense in X for every $t>0$ and $\lambda\in\mathbb{C}$ or there is a closed hyperplane H in X such that $T_t(H)\subseteq H$ for every $t\in\mathbb{R}_+$.*

The most of the section is devoted to the proof of Proposition 2.1. We need several elementary lemmas. Recall that a subset B of a vector space X is called *balanced* if $\lambda x\in B$ for every $x\in B$ and $\lambda\in\mathbb{K}$ such that $|\lambda|\leqslant 1$.

Lemma 2.2. *Let K be a compact subset of an infinite dimensional topological vector space and X such that $0\notin K$. Then $\Lambda=\{\lambda x:\lambda\in\mathbb{K},\ x\in K\}$ is a closed nowhere dense subset of X.*

Proof. Closeness of Λ in X is a straightforward exercise. Assume that Λ is not nowhere dense. Since Λ is closed, its interior L is non-empty. Since K is closed and $0\notin K$, we can find a non-empty balanced open set U such that $U\cap K=\varnothing$. Clearly $\lambda x\in L$ whenever $x\in L$ and $\lambda\in\mathbb{K}$, $\lambda\neq 0$. Since U is open and balanced the latter property of L implies that the open set $W=L\cap U$ is non-empty. Taking into account the definition of Λ, the inclusion $L\subseteq\Lambda$, the equality $U\cap K=\varnothing$ and the fact that U is balanced, we see that every $x\in W$ can be written as $x=\lambda y$, where $y\in K$ and $\lambda\in\mathbb{D}=\{z\in\mathbb{K}:|z|\leqslant 1\}$. Since both K and $\mathbb{D}$ are compact, $Q=\{\lambda y:\lambda\in\mathbb{D},\ y\in K\}$ is a compact subset of X. Since $W\subseteq Q$, W is a non-empty open set with compact closure. Such a set exists [7] only if X is finite dimensional. This contradiction completes the proof. ☐

The following lemma is a particular case of Lemma 5.1 in [6].

Lemma 2.3. *Let X be a complex topological vector space such that $2\leqslant \mathtt{dim}\,X<\infty$. Then X supports no supercyclic strongly continuous linear semigroups.*

Lemma 2.4. *Let X be an infinite dimensional topological vector space, $\lambda\in\mathbb{K}$, $t_0>0$ and $\{T_t\}_{t\in\mathbb{R}_+}$ be a strongly continuous linear semigroup such that $T_{t_0}=\lambda I$. Then $\{T_t\}_{t\in\mathbb{R}_+}$ is not supercyclic.*

Proof. Let $x\in X\setminus\{0\}$. It suffices to show that x is not a supercyclic vector for $\{T_t\}_{t\in\mathbb{R}_+}$.

First, we consider the case $\lambda=0$. By the strong continuity, there is $s>0$ such that $0\notin K=\{T_tx:t\in[0,s]\}$ and K is a compact subset of X. By Lemma 2.2, $A=\{zT_tx:z\in\mathbb{K},\ t\in[0,s]\}$ is nowhere dense in X. Take $n\in\mathbb{N}$ such that $ns\geqslant t_0$. Since $T_{t_0}=0$ and $ns\geqslant t_0$, we have $T_s^n=T_{ns}=0$. Then $Y=\overline{T_s(X)}\neq X$. In particular, Y is nowhere dense in X. Clearly, $T_tx\in Y$ whenever $t\geqslant s$. Hence $\{zT_tx:t\in\mathbb{R}_+,\ z\in\mathbb{K}\}$ is contained in $A\cup Y$ and therefore is nowhere dense in X. Thus x is not a supercyclic vector for $\{T_t\}_{t\in\mathbb{R}_+}$.

Assume now that $\lambda\neq 0$. Then $T_{t_0n}x=\lambda^nx\neq 0$ for every $n\in\mathbb{Z}_+$. Hence each of the compact sets $K_n=\{T_tx:t_0n\leqslant t\leqslant t_0(n+1)\}$ with $n\in\mathbb{Z}_+$ does not contain 0. By Lemma 2.2, the sets $A_n=\{zT_tx:z\in\mathbb{C},\ t_0n\leqslant t\leqslant t_0(n+1)\}$ are nowhere dense in X. On the other hand, for every $t\in[t_0n,t_0(n+1)]$, $T_{t+t_0}x=T_tT_{t_0}x=\lambda T_tx$ and therefore $A_n=A_{n+1}$ for each $n\in\mathbb{Z}_+$. Hence $\{zT_tx:t\in\mathbb{R}_+,\ z\in\mathbb{K}\}$, which is clearly the union of A_n, coincides with A_1 and therefore is nowhere dense. Thus x is not a supercyclic vector for $\{T_t\}_{t\in\mathbb{R}_+}$. ☐

Lemma 2.5. *Let X be a complex topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a supercyclic strongly continuous linear semigroup on X. Let also $t_0>0$ and $\lambda\in\mathbb{C}$. Then the space $Y=\overline{(T_{t_0}-\lambda I)(X)}$ either coincides with X or is a closed hyperplane in X.*

Proof. Using the semigroup property, it is easy to see that Y is invariant for each T_t. Factoring Y out, we arrive to a supercyclic strongly continuous linear semigroup $\{S_t\}_{t\in\mathbb{R}_+}$ acting on X/Y, where $S_t(x+Y)=T_tx+Y$. Obviously, $S_{t_0}=\lambda I$. If X/Y is infinite dimensional, we arrive to a contradiction with Lemma 2.4. If X/Y is finite dimensional and $\mathtt{dim}\,X/Y\geqslant 2$, we obtain a contradiction with Lemma 2.3. Thus $\mathtt{dim}\,X/Y\leqslant 1$, as required. □

Proof of Proposition 2.1. Assume that there is $t>0$ and $\lambda\in\mathbb{K}$ such that $(T_t-\lambda I)(X)$ is not dense in X. By Lemma 2.5, $H=\overline{(T_t-\lambda I)(X)}$ is a closed hyperplane in X. It is easy to see that H is invariant for every T_t. □

The following lemma provides some extra information on the second case in Proposition 2.1.

Lemma 2.6. *Let X be a complex topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a strongly continuous linear semigroup on X. Assume also that there is a closed hyperplane H in X such that $T_t(H)\subseteq H$ for every $t\in\mathbb{R}_+$ and let $f\in X'$ be such that $H=\mathtt{ker}\,f$. Then there exists $w\in\mathbb{C}$ such that $e^{wt}T'_tf=f$ for every $t\in\mathbb{R}_+$.*

Proof. Since $H=\mathtt{ker}\,f$ is invariant for every T_t, there is a unique function $\varphi:\mathbb{R}_+\to\mathbb{C}$ such that $T'_tf=\varphi(t)f$ for every $t\in\mathbb{R}_+$. Pick $u\in X$ such that $f(u)=1$. Then $(T'_tf)(u)=f(T_tu)=\varphi(t)$ for every $t\in\mathbb{R}_+$. Since $\{T_t\}_{t\in\mathbb{R}_+}$ is strongly continuous, φ is continuous. The semigroup property for $\{T_t\}_{t\in\mathbb{R}_+}$ implies the semigroup property for the dual operators: $T'_0=I$ and $T'_{t+s}=T'_tT'_s$ for every $t,s\in\mathbb{R}_+$. Together with the equality $T'_tf=\varphi(t)f$, it implies that $\varphi(0)=1$ and $\varphi(t+s)=\varphi(t)\varphi(s)$ for every $t,s\in\mathbb{R}_+$. The latter and the continuity of φ means that there is $w\in\mathbb{C}$ such that $\varphi(t)=e^{-wt}$ for each $t\in\mathbb{R}_+$. Thus $e^{wt}T'_tf=f$ for $t\in\mathbb{R}_+$, as required. □

3 Supercyclicity versus universality of affine maps

In this section we relate the supercyclicity of an operator or a semigroup in the case of the existence of an invariant hyperplane and the universality of an affine map or an affine semigroup. We start with the following general lemma.

Lemma 3.1. *Let X be a topological vector space, $u\in X$, $f\in X'\setminus\{0\}$, $f(u)=1$ and $H=\mathtt{ker}\,f$. Assume also that $\{T_a\}_{a\in A}$ is a family of continuous linear operators on X such that $T'_af=f$ for each $a\in A$. Then the family $\mathcal{F}=\{zT_a:z\in\mathbb{K},\ a\in A\}$ is universal if and only if the family $\mathcal{G}=\{R_a\}_{a\in A}$ of affine maps $R_a:H\to H$, $R_ax=(T_au-u)+T_ax$ is universal on H. Moreover, $x\in X$ is universal for $\mathcal{F}$ if and only if $x=\lambda(u+w)$, where $\lambda\in\mathbb{K}\setminus\{0\}$ and w is universal for $\mathcal{G}$. Next, if $A=\mathbb{Z}_+$ and $T_a=T_1^a$ for every $a\in\mathbb{Z}_+$, then $R_a=R_1^a$ for every $a\in\mathbb{Z}_+$. Finally, if $A=\mathbb{R}_+$ and $\{T_a\}_{a\in\mathbb{R}_+}$ is a strongly (respectively, jointly) continuous linear semigroup, then $\{R_a\}_{a\in\mathbb{R}_+}$ is a strongly (respectively, jointly) continuous affine semigroup.*

Proof. Since $T_a(H)\subseteq H$ for every a, vectors from H can not be universal for $\mathcal{F}$. Obviously, they also do not have the form $\lambda(u+w)$ with $\lambda\in\mathbb{K}\setminus\{0\}$ and $w\in H$.

Now let $x_0\in X\setminus H$. Then $f(x_0)\neq 0$ and therefore $x=\frac{x_0}{f(x_0)}\in u+H$. Since $T_a(u+H)\subseteq u+H$ for every $a\in A$, $O=\{T_ax:a\in A\}\subseteq u+H$. It is straightforward to see that x_0 is universal for $\mathcal{F}$ if and only if O is dense in $u+H$. That is, x_0 is universal for $\mathcal{F}$ if and only if x is universal for the family $\{Q_a\}_{a\in A}$, where each $Q_a:u+H\to u+H$ is the restriction of T_a to the invariant subset $u+H$. Obviously, the translation map $\Phi:H\to u+H$, $\Phi(y)=u+y$ is a homeomorphism and $R_a=\Phi^{-1}Q_a\Phi$ for every $a\in A$. It follows that x_0 is universal for $\mathcal{F}$ if and only if $\Phi^{-1}x=x-u$ is

universal for $\mathcal{G}$. Denoting $w = x - u$, we see that the latter happens if and only if $x_0 = f(x_0)(u+w)$ with $w \in \mathcal{U}(\mathcal{G})$.

Since Q_a are the restrictions of T_a to the invariant subset $u + H$ and R_a are similar to Q_a with the similarity independent on a, $\{R_a\}$ inherits all the semigroup or continuity properties from $\{T_a\}$. The proof is complete. $\square$

The following two lemmas are particular cases of Lemma 3.1.

Lemma 3.2. *Let X be a topological vector space, $u \in X$, $f \in X' \setminus \{0\}$, $f(u) = 0$ and $H = \mathtt{ker}\, f$. Then $T \in L(X)$ satisfying $T'f = f$ is supercyclic if and only if the map $R : H \to H$, $Rx = (Tu - u) + Tx$ is universal. Moreover, $x \in X$ is a supercyclic vector for T if and only if $x = \lambda(u + w)$, where $\lambda \in \mathbb{K} \setminus \{0\}$ and $w \in \mathcal{U}(R)$.*

Lemma 3.3. *Let X be a topological vector space, $u \in X$, $f \in X' \setminus \{0\}$, $f(u) = 1$ and $H = \mathtt{ker}\, f$. Then a strongly (respectively, jointly) continuous linear semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ on X satisfying $T_t'f = f$ for $t \in \mathbb{R}_+$ is supercyclic if and only if the strongly (respectively, jointly) continuous affine semigroup $\{R_t\}_{t\in\mathbb{R}_+}$ on H defined by $R_t x = (T_t u - u) + T_t x$ is universal. Moreover, $x \in X$ is a supercyclic vector for $\{T_t\}_{t\in\mathbb{R}_+}$ if and only if $x = \lambda(u+w)$, where $\lambda \in \mathbb{K}\setminus\{0\}$ and $w \in \mathcal{U}(\{R_t\}_{t\in\mathbb{R}_+})$.*

4 Universality of affine semigroups

The proof of the following lemma is a matter of an easy routine verification.

Lemma 4.1. *Let X be a topological vector space, $\{T_t\}_{t\in\mathbb{R}_+}$ be a collection of continuous affine maps on X, $\{S_t\}_{t\in\mathbb{R}_+}$ be a collection of continuous linear operators on X and $t \mapsto w_t$ be a map from $\mathbb{R}_+$ to X such that $T_t x = w_t + S_t x$ for every $t \in \mathbb{R}_+$ and $x \in X$.*

Then $\{T_t\}_{t\in\mathbb{R}_+}$ is an affine semigroup if and only if $\{S_t\}_{t\in\mathbb{R}_+}$ is a linear semigroup,

$$w_0 = 0 \ \text{and} \quad w_{t+s} = w_t + S_t w_s \quad \text{for every } s, t \in \mathbb{R}_+. \tag{4.1}$$

Moreover, the semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ is strongly continuous if and only if $\{S_t\}_{t\in\mathbb{R}_+}$ is strongly continuous and the map $t \mapsto w_t$ is continuous. Finally, the semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ is jointly continuous if and only if $\{S_t\}_{t\in\mathbb{R}_+}$ is jointly continuous and the map $t \mapsto w_t$ is continuous.

Lemma 4.2. *Let X be a topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a universal strongly continuous affine semigroup on X. Then $(I - T_t)(X)$ is dense in X for every $t > 0$.*

Proof. Assume the contrary. Then there is $s > 0$ such that $Y_0 \neq X$, where $Y_0 = \overline{(I - T_s)(X)}$. Let Y be a translation of Y_0, containing 0: $Y = Y_0 - u_0$ with $u_0 \in Y_0$. it is easy to see that, factoring out the closed linear subspace Y, we arrive to the universal strongly continuous affine semigroup $\{F_t\}_{t\in\mathbb{R}_+}$ on X/Y, where $F_t(x+Y) = T_t x + Y$ for every $t \in \mathbb{R}_+$ and $x \in X$. By definition of Y, the linear part of F_s is I. Let $\alpha \in X/Y$ be a universal vector for $\{F_t\}_{t\in\mathbb{R}_+}$. By Lemma 4.1, there is a strongly continuous linear semigroup $\{G_t\}_{t\in\mathbb{R}_+}$ on X/Y and a continuous map $t \mapsto \gamma_t$ from $\mathbb{R}_+$ to X/Y such that $\gamma_0 = 0$, $F_t\beta = G_t\beta + \gamma_t$ and $\gamma_{r+t} = \gamma_r + G_r\gamma_t = \gamma_t + G_t\gamma_r$ for every $\beta \in X/Y$ and $r, t \in \mathbb{R}_+$. Using these relations and the equality $G_s = I$, we obtain that $F_{t+ns}\alpha = F_t\alpha + n\gamma_s$ for every $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$. It follows that

$$\{F_t\alpha : t \in \mathbb{R}_+\} = K + \mathbb{Z}_+\gamma_s, \quad \text{where } K = \{F_t\alpha : t \in [0, s]\}.$$

Since α is universal for $\{F_t\}_{t\in\mathbb{R}_+}$, by the last display, $O = K + \mathbb{Z}_+\gamma_s$ is dense in X/Y. Since O is closed as a sum of a compact set and a closed set, $O = X/Y$. On the other hand, O does not contain $-c\gamma_s$ for any sufficiently large $c > 0$. This contradiction completes the proof. $\square$

Lemma 4.3. *Let X be a topological vector space, $x \in X$, $s > 0$ and $\{T_t\}_{t\in\mathbb{R}_+}$ be a universal affine semigroup on X. Assume also that $T_t x = S_t x + w_t$, where $\{S_t\}_{t\in\mathbb{R}_+}$ strongly continuous linear semigroup on X and $t \mapsto w_t$ is a continuous map from $\mathbb{R}_+$ to X. Then $\{S_t\}_{t\in\mathbb{R}_+}$ is hypercyclic. Moreover, $\mathcal{U}(\{S_t\}_{t\in\mathbb{R}_+}) \cap (w_s + (I - S_s)(X)) \neq \varnothing$ for every $s > 0$.*

Proof. Let $x \in \mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$ and fix $s > 0$. By Lemma 4.2, $(T_s - I)(X)$ is dense in X. Hence $O = \{(T_s - I)T_t x : t \in \mathbb{R}_+\}$ is dense in X. Using the semigroup property of $\{T_t\}_{t\in\mathbb{R}_+}$ and $\{S_t\}_{t\in\mathbb{R}_+}$ together with (4.1), we get

$$(T_s - I)T_t x = S_s S_t x + S_s w_t + w_s - S_t x - w_t = S_t S_s x + S_t w_s - S_t x = S_t(w_s - (I - S_s)x)$$

for every $t \in \mathbb{R}_+$. By the above display, O is exactly the S_t-orbit of $w_s - (I - S_s)x$. Since O is dense in X, $w_s - (I - S_s)x \in w_s + (I - S_s)(X)$ is a hypercyclic vector for $\{S_t\}_{t\in\mathbb{R}_+}$ and therefore $\mathcal{U}(\{S_t\}_{t\in\mathbb{R}_+}) \cap (w_s + (I - S_s)(X)) \neq \varnothing$. □

Lemma 4.4. *Let X be a topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be an affine semigroup on X. Then for every $t_1, \ldots, t_n \in \mathbb{R}_+$ and every $z_1, \ldots, z_n \in \mathbb{K}$ satisfying $z_1 + \ldots + z_n = 1$, the map $S = z_1 T_{t_1} + \ldots + z_n T_{t_n}$ commutes with every T_t.*

Proof. It is easy to verify that for every affine map $A : X \to X$ and every $x_1, \ldots, x_n \in X$,

$$A(z_1 x_1 + \ldots + z_n x_n) = z_1 A x_1 + \ldots + z_n A x_n \ \text{ provided } z_j \in \mathbb{K} \text{ and } z_1 + \ldots + z_n = 1.$$

Let $t \in \mathbb{R}_+$. By the above display, $T_t S = z_1 T_t T_{t_1} + \ldots + z_n T_t T_{t_n}$. Since T_τ commute with each other, we get $T_t S = z_1 T_{t_1} T_t + \ldots + z_n T_{t_n} T_t = S T_t$. □

Lemma 4.5. *Let X be a topological vector space, $\{T_t\}_{t\in\mathbb{R}_+}$ be a universal strongly continuous affine semigroup on X and $x \in \mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$. Then $\Lambda(x) \subseteq \mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$, where*

$$\Lambda(x) = \{z_1 T_{t_1} x + \ldots + z_n T_{t_n} x : n \in \mathbb{N},\ t_j \in \mathbb{R}_+,\ z_j \in \mathbb{K},\ z_1 + \ldots + z_n = 1\}. \tag{4.2}$$

Proof. Let $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}_+$, $z_1, \ldots, z_n \in \mathbb{K}$ and $z_1 + \ldots + z_n = 1$. We have to show that $Ax \in \mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$, where $A = z_1 T_{t_1} + \ldots + z_n T_{t_n}$. By Lemma 4.4, A commutes with each T_t. Since $x \in \mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$, it suffices to verify that $A(X)$ is dense in X. By Lemma 4.1, we can write $T_t y = S_t y + w_t$ for every $y \in X$, where $\{S_t\}_{t\in\mathbb{R}_+}$ is a strongly continuous linear semigroup on X and $t \mapsto w_t$ is a continuous map from $\mathbb{R}_+$ to X. By Lemma 4.3, $\{S_t\}_{t\in\mathbb{R}_+}$ is hypercyclic. As shown in [3], every non-trivial linear combination of members of a hypercyclic strongly continuous linear semigroup has dense range. Thus $B = z_1 S_{t_1} + \ldots + z_n S_{t_n}$ has dense range. Since $A(X)$ is a translation of $B(X)$, $A(X)$ is also dense in X, which completes the proof. □

Proof of Theorem 1.3. Let X be a topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a universal jointly continuous affine semigroup on X. Lemmas 4.1 and 4.3 provide a hypercyclic jointly continuous linear semigroup on X. By Theorem A, there is a hypercyclic continuous linear operator on X. Since no such thing exists on a finite dimensional topological vector space [8], X is infinite dimensional. Since any compact subspace of an infinite dimensional topological vector space is nowhere dense [7], condition (1) of Proposition 1.1 is satisfied. Now let $x \in \mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$. By Lemma 4.5, the set $\Lambda(x)$ defined in (4.2) consists entirely of universal vectors for $\{T_t\}_{t\in\mathbb{R}_+}$. Clearly, $\{T_t x : t \in \mathbb{R}_+\} \subseteq \Lambda(x)$. By its definition, $\Lambda(x)$ is an affine subspace (=a translation of a linear subspace) of X. Since every affine subspace of a topological vector space is connected, locally path connected and simply connected, $\Lambda(x)$ satisfies all requirements for the set $Y_{c,x}$ (for every $c > 0$) from condition (2) in Proposition 1.1. By Proposition 1.1, $\mathcal{U}(T_s) = \mathcal{U}(\{T_t\}_{t\in\mathbb{R}_+})$ for every $s > 0$, as required. □

5 Proof of Theorem 1.2

Let X be a complex topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be a supercyclic jointly continuous linear semigroup on X. We have to prove that each T_s with $s>0$ is supercyclic and the set of supercyclic vectors of T_s coincides with the set of supercyclic vectors of $\{T_t\}_{t\in\mathbb{R}_+}$. If $T_t-\lambda I$ has dense range for every $t>0$ and every $\lambda\in\mathbb{C}$, then Proposition C provides the required result. Otherwise, by Proposition 2.1, there is a closed hyperplane H in X invariant for every T_t. By Lemma 2.6, there are $f\in X'$ and $\alpha\in\mathbb{C}$ such that $H=\mathtt{ker}\,f$ and $e^{\alpha t}T_t'f=f$ for every $t\in\mathbb{R}_+$. Clearly $\{e^{\alpha t}T_t\}_{t\in\mathbb{R}_+}$ is a jointly continuous supercyclic linear semigroup on X with the same set $\mathcal{S}$ of supercyclic vectors as the original semigroup $\{T_t\}_{t\in\mathbb{R}_+}$. Fix $u\in X$ satisfying $f(u)=1$. Now fix $s>0$ and $v\in\mathcal{S}$. We have to show that v is supercyclic for T_s. By Lemma 3.3, applied to the semigroup $\{e^{\alpha t}T_t\}_{t\in\mathbb{R}_+}$, we can write $v=\lambda(u+y)$, where $\lambda\in\mathbb{K}\setminus\{0\}$ and y is a universal vector for the jointly continuous affine semigroup $\{R_t\}_{t\in\mathbb{R}_+}$ on H defined by the formula $R_tx=w_t+e^{\alpha t}T_tx$ with $w_t=(e^{\alpha t}T_t-I)u$. By Theorem 1.3, y is universal for R_s. By Lemma 3.2, $v=\lambda(u+y)$ is a supercyclic vector for $e^{\alpha s}T_s$ and therefore v is a supercyclic vector for T_s. The proof is complete.

6 Remarks

By Lemma 4.3, universality of a strongly continuous affine semigroup implies hypercyclicity of the underlying linear semigroup. The following example shows that that the converse is not true.

Example 6.1. Consider the backward weighted shift $T\in L(\ell_2)$ with the weight sequence $\{e^{-2n}\}_{n\in\mathbb{N}}$. That is, $Te_0=0$ and $Te_n=e^{-2n}e_{n-1}$ for $n\in\mathbb{N}$, where $\{e_n\}_{n\in\mathbb{Z}_+}$ is the standard basis of ℓ_2. Then the jointly continuous linear semigroup $\{S_t\}_{t\in\mathbb{R}_+}$ with $S_t=e^{t\ln(I+T)}$ is hypercyclic. Moreover, there exists a continuous map $t\mapsto w_t$ from $\mathbb{R}_+$ to ℓ_2 such that $\{T_t\}_{t\in\mathbb{R}_+}$ is a jointly continuous non-universal affine semigroup, where $T_tx=w_t+S_tx$ for $x\in\ell_2$.

Proof. Since T, being a compact weighted backward shift, is quasinilpotent, the operator $\ln(I+T)$ is well defined and bounded and $\{S_t\}_{t\in\mathbb{R}_+}$ is a jointly continuous linear semigroup. Moreover, $S_1=I+T$ is hypercyclic according to Salas [4] as a sum of the identity operator and a backward weighted shift. Hence $\{S_t\}_{t\in\mathbb{R}_+}$ is hypercyclic.

Let $u\in\ell_2$, $u_n=(n+1)^{-1}$ for $n\in\mathbb{Z}_+$. For each $t\in\mathbb{R}_+$, let $w_t=\nu_t(T)u$, where $\nu_s(z)=\sum\limits_{n=1}^{\infty}\frac{s(s-1)\dots(s-n+1)}{n!}z^{n-1}$. Since T is quasinilpotent, $\nu_t(T)$ are well defined bounded linear operators and the map $t\mapsto\nu_t(T)$ is operator-norm continuous. Hence $t\mapsto w_t$ is continuous as a map from $\mathbb{R}_+$ to ℓ_2. It is easy to verify that $w_0=0$, $w_1=u$ and $w_{t+s}=S_tw_s+w_t$ for every $s,t\geqslant 0$. By Lemma 4.1, $\{T_t\}_{t\in\mathbb{R}_+}$ is a jointly continuous affine semigroup, where $T_tx=w_t+S_tx$. It remains to show that $\{T_t\}_{t\in\mathbb{R}_+}$ is non-universal. Assume the contrary. Since $w_1=u$ and $S_1=I+T$, Lemma 4.3 implies that the coset $u+T(\ell_2)$ must contain a hypercyclic vector for $I+T$. This however is not the case as shown in [5, Proposition 7.16]. □

Recall that a topological space X is called a *Baire space* if the intersection of any countable collection of dense open subsets of X is dense in X.

Remark 6.2. Let X be a topological vector space and $S\in L(X)$ be hypercyclic. If $u\in(I-S)(X)$, then the affine map $Tx=u+Sx$ is universal. Indeed, let $w\in X$ be such that $u=w-Sw$. It is easy to show that $T^nx=w+S^n(x-w)$ for every $x\in X$ and $n\in\mathbb{N}$. Thus x is universal for T if and only if $x-w$ is universal for S.

If additionally X is separable metrizable and Baire, then a standard Baire category type argument shows that the set of $u\in X$ for which the affine map $Tx=u+Sx$ is universal is a dense G_δ-subset of X. Example 6.1 shows that this set can differ from X.

Recall that a locally convex topological vector space X is called *barrelled* if every closed convex balanced subset B of X satisfying $X = \bigcup_{n=1}^{\infty} nB$ contains a neighborhood of 0. As we have already mentioned in the introduction, the joint continuity of a linear semigroup follows from the strong continuity if the underlying space X is an $\mathcal{F}$-space. The same is true for wider classes of topological vector spaces. For instance, it is sufficient for X to be a Baire topological vector space or a barrelled locally convex topological vector space [7]. Thus the following observation holds true.

Remark 6.3. The joint continuity condition in Theorems A, 1.2 and 1.3 can be replaced by the strong continuity, provided X is Baire or X is locally convex and barrelled.

For general topological vector spaces however strong continuity of a linear semigroup does not imply joint continuity. Moreover, the following example shows that Theorem A fails in general if the joint continuity condition is replaced by the strong continuity. Recall that the Fréchet space $L^2_{\rm loc}(\mathbb{R}_+)$ consists of the (equivalence classes of) scalar valued functions $\mathbb{R}_+$, square integrable on $[0,c]$ for each $c>0$. Its dual space can be naturally interpreted as the space $L^2_{\rm fin}(\mathbb{R}_+)$ of (equivalence classes of) square integrable scalar valued functions $\mathbb{R}_+$ with bounded support. The duality between $L^2_{\rm loc}(\mathbb{R}_+)$ and $L^2_{\rm fin}(\mathbb{R}_+)$ is provided by the natural dual pairing $\langle f,g\rangle = \int_0^\infty f(t)g(t)\,dt$. Obviously the linear semigroup $\{S_t\}_{t\in\mathbb{R}_+}$ of backward shifts $S_tf(x) = f(x+t)$ is strongly continuous and therefore jointly continuous on the Fréchet space $L^2_{\rm loc}(\mathbb{R}_+)$. It follows that the same semigroup is strongly continuous on $L^2_{\sigma,{\rm loc}}(\mathbb{R}_+)$ being $L^2_{\rm loc}(\mathbb{R}_+)$ endowed with the weak topology.

Example 6.4. Let $X = L^2_{\sigma,{\rm loc}}(\mathbb{R}_+)$ and $\{S_t\}_{t\in\mathbb{R}_+}$ be the above strongly continuous semigroup on X. Then there is $f\in X$ hypercyclic for $\{S_t\}_{t\in\mathbb{R}_+}$ such that f is non-hypercyclic for S_1.

Proof. Let H be the hyperplane in $L^2[0,1]$ consisting of the functions with zero Lebesgue integral. Fix a norm-dense countable subset A of H. One can easily construct $f\in L^2_{\rm loc}(\mathbb{R}_+)$ such that

(a) for every $n\in\mathbb{N}$, the function $f_n:[0,1]\to\mathbb{K}$, $f_n(t)=f(n+t)$ belongs to A;

(b) for every $n\in\mathbb{N}$ and $h_1,\dots,h_n\in A$, there is $m\in\mathbb{N}$ such that $h_j=f_{m+j}$ for $1\leqslant j\leqslant n$.

For $s\in\mathbb{R}_+$, let $\chi_s\in X'=L^2_{\rm fin}(\mathbb{R}_+)$ be the indicator function of the interval $[s,s+1]$: $\chi_s(t)=1$ if $s\leqslant t\leqslant s+1$ and $\chi_s(t)=0$ otherwise. By (a), $S_1^nf\in \texttt{ker}\,\chi_0$ for every $n\in\mathbb{N}$ and therefore f is not a hypercyclic vector for S_1.

It remains to show that f is a hypercyclic vector for $\{S_t\}_{\in\mathbb{R}_+}$ acting on X. Using (a) and (b), we see that the Fréchet space topology closure of the orbit $\{S_tf:t\in\mathbb{R}_+\}$ is exactly the set

$$O=\bigcup_{s\in[0,1)}\bigcap_{n\in\mathbb{Z}_+}\texttt{ker}\,\chi_{s+n}.$$

In order to show that f is hypercyclic for $\{S_t\}_{\in\mathbb{R}_+}$ acting on X, it suffices to verify that O is dense in $L^2_{\sigma,{\rm loc}}(\mathbb{R}_+)$. Assume the contrary. Then there is a weakly open set W in $L^2_{\rm loc}(\mathbb{R}_+)$, which does dot intersect O. That is, there are linearly independent $\varphi_1,\dots,\varphi_m\in L^2_{\rm fin}(\mathbb{R}_+)$ and $c_1,\dots,c_m\in\mathbb{K}$ such that

$$\max_{1\leqslant j\leqslant m}|c_j-\langle g,\varphi_j\rangle|\geqslant 1 \text{ for every } g\in O.$$

Let $k\in\mathbb{N}$ be such that each φ_j vanishes on $[k,\infty)$. Pick any $0<t_0<\dots<t_m<1$. Note that for every $l\in\{0,\dots,m\}$, the restrictions of the functionals φ_j to $\bigcap_{n=0}^{k}\texttt{ker}\,\chi_{t_l+n}$ are not linearly independent. Indeed, otherwise we can find $h_0\in\bigcap_{n=0}^{k}\texttt{ker}\,\chi_{t_l+n}$ such that $\langle h_0,\varphi_j\rangle=c_j$ for $1\leqslant j\leqslant m$. It is easy to see that there is $h\in L^2_{\rm loc}(\mathbb{R}_+)$ such that $h\big|_{[0,k]}=h_0\big|_{[0,k]}$, $h\big|_{[k+1,\infty)}=0$

and $\langle h,\chi_{t_l+k-1}\rangle=\langle h,\chi_{t_l+k}\rangle=0$. Then $\langle h,\varphi_j\rangle=c_j$ for $1\leqslant j\leqslant m$ and $h\in\bigcap\limits_{n=0}^{\infty}\texttt{ker}\,\chi_{t_l+n}\subseteq O$. We have arrived to a contradiction with the above display.

The fact that φ_j are not linearly independent on $\bigcap\limits_{n=0}^{k}\texttt{ker}\,\chi_{t_l+n}$ implies that there is a non-zero $g_l\in\texttt{span}\,\{\varphi_1,\dots,\varphi_m\}\cap\texttt{span}\,\{\chi_{t_l},\dots,\chi_{t_l+k}\}$. Since χ_{t_l+r} are all linearly independent, $g_0,\dots,g_m$ are $m+1$ linearly independent vectors in the m-dimensional space $\texttt{span}\,\{\varphi_1,\dots,\varphi_m\}$. This contradiction completes the proof. $\square$

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